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Time-Dependent 2D Harmonic Oscillator in Presence of the Aharanov-Bohm Effect

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We use the Lewis-Riesenfeld theory to determine the exact form of the wavefunctions of a two-dimensionnal harmonic oscillator with time-dependent mass and frequency in presence of the Aharonov-Bohm effect (AB). We find that the auxiliary equation is independent of the AB magnetic flux. In the particular case of quantized AB magnetic flux the wavefunctions coincide exactly with the wavefunctions of the 2D time-dependent harmonic oscillator.

KEY WORDS: harmonic oscillator; magnetic field; Aharonov-Bohm effect; exact wave function.

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In the last few decades the problem of time-dependent systems have played a major role in the study of several physics phenomena (Chung-In *et al.*, 2002; Kleber, 1994; Markov, 1989). A great deal of attention has been paid to some specific problems of time-dependent oscillators among them the time-dependent singular oscillator. In fact this specific problem has been studied extensively in different direction by many authors by whom closed-form solutions are obtained in explicit form (Dodonov *et al.*, 1974; Maamache, 1995, 1996, 2000; Maamache and Bekkar, 2003; Malkin and Man'ko, 1972; Pedrosa *et al.*, 1997; Trifonov, 1999). The construction of the invariant (Lewis and Riesenfeld, 1969) (constants of the motion), has attracted much attention, which describe a quantum system governed by a time-dependent Hamiltonian. Lewis and Riensenfeld (Lewis and Riesenfeld, 1969) have shown that, if the system admits an invariant $I(t)$, it is possible to find a privileged basis of eigenstates of this operator when multiplied

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by suitable time-dependent phase factor, evolve according to the time Schrodinger equation.

In the meantime the problem of two-dimensional (2D) systems as well as the 2D time-dependent harmonic oscillators in the presence of a magnetic field (Abdalla, 1998; Baseia *et al.* 1992; Bassalo *et al.*, 1989; Choi, 2004; Dodonov *et al.*, 1976; Ferreira *et al.*, 2002; Liang and Zhang, 2003; Maamache *et al.*, in press; Malkin and Manko, 1970a,b; Nassar, 1987; Yuce, 2003) has been considered, for which the dynamical operators and the wave function have been obtained.

On the other hand, the Aharonov-Bohm (AB) effect (Aharonov and Bohm, 1959) i.e. systems in which charged particle interact with the vector potential of an infinitely long thin magnetic string (AB potential) are still receiving considerable interest with applications in various area in the literature (Azevedo, 2003; Ferkous and Bounames, 2004; Furtado and Moraes, 2000; Hagen, 1990, 1991, 1993; Peshkin and Tonomura, 1989; Zhu and Henneberger, 1993). In such systems the nonlocal character of the interaction of the charged particle with the magnetic field of the string leads, quantum mechanically, to observable physical effect despite the absence of Lorentz forces on the particle.

In the present work, we consider a two-dimensional (2D) harmonic oscillator with time-dependent mass and frequency confined to the (x, y) plane, with a whisker of flux (a very thin solenoid) ν on the z-axis in the positive direction. The time-dependent Schrodinger equation is given by

$$i\hbar\frac{\partial}{\partial t}\psi = \left[\frac{(\mathbf{p}-e\mathbf{A})^2}{2M(t)} + \frac{1}{2}M(t)\omega^2(t)(x^2+y^2)\right]\psi, \tag{1}$$

where $\mathbf{A}$ is the potential vector. The magnetic field $\mathbf{B}$, associated to the AB effect, is assumed to be perpendicular to the plane and confined to a thin magnetized filament

$$eB(r) = -\frac{\nu}{r}\delta(r), \tag{2}$$

where ν is a finite and nonzero flux parameter.

It is well known (Hagen, 1990, 1991, 1993) that for a flux tube of zero radius the corresponding form of the potential $\mathbf{A}$ in the coulomb gauge is

$$e\mathbf{A} = \left(\frac{\nu y}{x^2+y^2}, -\frac{\nu x}{x^2+y^2}, 0\right).$$

Then the time-dependent Schrodinger equation (1) becomes

$$i\hbar\frac{\partial}{\partial t}\psi = H_\nu(t)\psi,$$

where the Hamiltonian $H_\nu(t)$ is given by

$$H_\nu(t) = \frac{p_x^2 + p_y^2}{2M(t)} + \frac{1}{2}M(t)\omega^2(t)(x^2 + y^2) + \frac{\nu L_z}{M(t)(x^2 + y^2)} + \frac{\nu^2}{2M(t)(x^2 + y^2)}, \tag{3}$$

and $L_z = \left(xp_y - yp_x\right)$ is the angular momentum.

For the construction of an exact invariant for the quantum system described by the time-dependent Hamiltonian (3), we use the Lie algebraic approach (Maamache, 1995). Let us introduce the Hermitian basis

$$T_1^\nu = \frac{1}{2}\left[p_x^2 + p_y^2 + \frac{2\nu L_z}{x^2 + y^2} + \frac{\nu^2}{x^2 + y^2}\right],$$

$$T_2 = \frac{1}{2}(p_x x + x p_x + y p_y + p_y y),$$

$$T_3 = \frac{1}{2}(x^2 + y^2),$$

which is closed with respect to

$$\left[T_1^\nu, T_2\right] = -2i\hbar T_1^\nu, \quad [T_2, T_3] = -2i\hbar T_3, \quad \left[T_1^\nu, T_3\right] = -i\hbar T_2. \tag{4}$$

We note that the above Lie algebra $\left\{T_1^\nu, T_2, T_3\right\}$ is identical to the 2D oscillator algebra $\left\{T_1^{\nu=0}, T_2, T_3\right\}$ for the particular case $\nu = 0$.

Now, we look for the invariant in the form

$$I(t) = \mu_1(t)T_1^\nu + \mu_2(t)T_2 + \mu_3(t)T_3, \tag{5}$$

and by means of $\frac{\partial I}{\partial t} = \frac{i}{\hbar}[I, H]$ and comparison of the coefficients of a system of first-order linear differential equation for the unknown μ_r in (5) is obtained

$$\dot{\mu}_1 = -\frac{2}{M}\mu_2, \tag{6}$$

$$\dot{\mu}_2 = M\omega^2\mu_1 - \frac{1}{M}\mu_3, \tag{7}$$

$$\dot{\mu}_3 = 2M\omega^2\mu_2, \tag{8}$$

which can be simplified by setting $\mu_1 = \rho^2$ where ρ is the solution of the auxiliary equation

$$\ddot{\rho} + \frac{\dot{M}}{M}\dot{\rho} + \rho\omega^2 = \frac{1}{M^2\rho^3}, \tag{9}$$

and the other coefficients are $\mu_2 = -M\rho\dot{\rho}$, and $\mu_3 = \frac{1}{\rho^2}(1 + M^2\rho^2\dot{\rho}^2)$. We note that the above auxiliary equation does not depend on the magnetic flux ν.

Thus, the invariant (5) can be written in the form

$$I\left(t\right)=\frac{1}{2}\left\{\rho^{2}\left(p_{x}^{2}+p_{y}^{2}+\frac{2\nu L_{z}}{x^{2}+y^{2}}+\frac{\nu^{2}}{x^{2}+y^{2}}\right)-M\rho\dot{\rho}\left(p_{x}x+xp_{x}\right.\right.$$
$$\left.\left.+yp_{y}+p_{y}y\right)+\frac{1}{\rho^{2}}\left(1+M^{2}\rho^{2}\dot{\rho}^{2}\right)\left(x^{2}+y^{2}\right)\right\}. \qquad (10)$$

According to the Lewis-Riesenfeld theory (Lewis and Riesenfeld, 1969), given a physical system that contains an invariant operator $I(t)$, the following results can be obtained:

(a) its eigenvalues $\lambda_{n,m}$ are time-independent,

$$I\phi_{n,m}\left(x,y,t\right)=\lambda_{n,m}\phi_{n,m}\left(x,y,t\right), \qquad (11)$$

(b) its eigenfunctions $\phi_{n,m}\left(x,y,t\right)$ are time-dependent and if multiplied by suitable phases such as $\exp[i\alpha_{n,m}(t)]$, with the $\alpha_{n,m}(t)$ verifying

$$\hbar\dot{\alpha}_{n,m}(t)=\left\langle\phi_{n,m}\right|i\hbar\frac{\partial}{\partial t}-H(t)\left|\phi_{n,m}\right\rangle, \qquad (12)$$

then, the wavefunctions $\psi_{n,m}(x,y,t)=\exp[i\alpha_{n,m}(t)]\phi_{n,m}(x,y,t)$ evolve according to the time-dependent Shrodinger equation. The general solution $\psi(x,y,t)$ can then be written as

$$\psi(x,y,t)=\sum_{n,m}C_{n,m}\psi_{n,m}(x,y,t), \qquad (13)$$

where $C_{n,m}$ are arbitrary constants coefficients fixed by the initial conditions of the physical system.

Let us consider the unitary transformation

$$\phi'_{n,m}(x,y,t)=U\phi_{n,m}(x,y,t), \qquad (14)$$

where

$$U=\exp\left[-\frac{iM\dot{\rho}}{2\hbar\rho}(x^{2}+y^{2})\right]. \qquad (15)$$

Under this unitary transformation the eigenvalue equation (11) is mapped into

$$I'\phi'_{n,m}\left(x,y,t\right)=\lambda_{n,m}\phi'_{n,m}\left(x,y,t\right), \qquad (16)$$

with

$$I'=UIU^{+}=\frac{1}{2}\left[\rho^{2}\left(p_{x}^{2}+p_{y}^{2}\right)+\frac{2\nu\rho^{2}}{x^{2}+y^{2}}L_{z}+\frac{\nu^{2}\rho^{2}}{x^{2}+y^{2}}+\frac{x^{2}+y^{2}}{\rho^{2}}\right]. \qquad (17)$$

If one define the new variables $\sigma(t) = \frac{x}{\rho}$, $\eta(t) = \frac{y}{\rho}$ and in polar coordinates the eigenvalues equation (11) takes the form

$$\frac{1}{2}\left[-\hbar^2\left(\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\varphi^2}\right)+\frac{2\nu}{r^2}\frac{\hbar}{i}\frac{\partial}{\partial\varphi}+\frac{\nu^2}{r^2}+r^2\right]\chi_{n,m} = \lambda_{n,m}\chi_{n,m}, \tag{18}$$

where $r^2 = \sigma^2 + \eta^2$, $\varphi = \tan^{-1}\left(\frac{y}{x}\right)$ and

$$\phi'_{n,m}(x, y, t) = \frac{1}{\rho}\chi_{n,m}(\sigma, \eta), \tag{19}$$

the solution of the above Equation (18) is given by Flügge (1994)

$$\chi_{n,m}(r,\varphi) = A_{n,m}\exp\left[-\frac{r^2}{2\hbar}\right]r^{\left|m+\frac{\nu}{\hbar}\right|}{}_1F_1\left(-n, \left|m+\frac{\nu}{\hbar}\right|+1, \frac{r^2}{\hbar}\right)\exp\left[im\varphi\right]$$
$$+ B_{n,m}r^{-\left|m+\frac{\nu}{\hbar}\right|}{}_1F_1\left(-n-2\left|m+\frac{\nu}{\hbar}\right|, 1-\left|m+\frac{\nu}{\hbar}\right|, \frac{r^2}{\hbar}\right)\exp\left[im\varphi\right], \tag{20}$$

with the constant eigenvalues

$$\lambda_{n,m} = \hbar\left(2n+\left|m+\frac{\nu}{\hbar}\right|+1\right), n = 0, 1, 2, ..., m = 0, \pm 1, \pm 2, \tag{21}$$

that depends on the magnetic flux ν which breaks the degeneracy of the eigenvalues levels.

Since (20) contains a regular and irregular solutions, one could simply require that $\chi_{n,m}(r,\varphi)$ be finite at the origin $r = 0$ and thereby eliminate ab initio the irregular solution (Hagen, 1993). This, in fact, gives

$$\chi_{n,m}(r,\varphi) = A_{n,m}\exp\left[-\frac{r^2}{2\hbar}\right]r^{\left|m+\frac{\nu}{\hbar}\right|}{}_1F_1\left(-n, \left|m+\frac{\nu}{\hbar}\right|+1, \frac{r^2}{\hbar}\right)\exp\left[im\varphi\right]. \tag{22}$$

On the other hand, substituting the Hamiltonian (3) and Equation (14) into Equation (12), we find that the phase function $\alpha_{n,m}(t)$ is given by

$$\alpha_{n,m}(t) = -\left(2n+\left|m+\frac{\nu}{\hbar}\right|+1\right)\int_0^t\frac{dt'}{M(t')\rho^2}. \tag{23}$$

Finally, the exact solution of the Schrodinger Equation (1) in the polar coordinates is given by

$$\psi_{n,m}(r,\varphi,t) = \frac{A_{n,m}}{\rho}\exp\left[\frac{iM}{2\hbar}\left(\rho\dot{\rho}+\frac{i}{M}\right)r^2\right]\exp\left[-i\left(2n+\left|m+\frac{\nu}{\hbar}\right|+1\right)\right.$$
$$\left.\times\int_0^t\frac{dt'}{M(t')\rho^2}\right]r^{\left|m+\frac{\nu}{\hbar}\right|}{}_1F_1\left(-n, \left|m+\frac{\nu}{\hbar}\right|+1, \frac{r^2}{\hbar}\right)\exp\left[im\varphi\right]. \tag{24}$$

In the absence of the AB effect $\nu = 0$, the solution (24) reduces to the solution of the time-dependent 2D harmonic oscillator. On the other hand, we note that if the AB magnetic flux is quantized, $\frac{\nu}{\hbar} = integer$, the wavefunctions (24) coincide also exactly with the solution of the time-dependent 2D harmonic oscillator.

In conclusion, such model has in fact been presented in the contest of obtaining the solution corresponding to zero radius of filament containing the Aharonov-Bohm flux. On the other hand, if we replace the flux tube of zero radius by one of finite radius R, with the additional condition that the magnetic field is confined to the surface of the tube, we point that the part of the operator T_1^{ν} which depends on the space-coordinates must be multiplied by the step function $\theta\,(r - R)$. Consequently, the problem becomes complicated because the algebra based on T_1^{ν}, T_2 and T_3 is not closed and there exist a physical formulation (Hagen, 1990, 1991, 1993) which eliminates these mathematical difficulties. Work in this direction is in progress.

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